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Some Properties of Anti-inverse semi group in LA-semiring

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Abstract:- Authors determine some different additive and multiplicative structures and congruence of anti-inverse semigroup of LA-semiring which satisfies the identity a + 1 = 1.

Keywords:- Anti-inverse semigroup, Seperative semigroup and congruence

I. DEFINITIONS

Definition1.1. For the elements x and y of a semigroup S, we say that they are mutually anti-inverse if the following conditions hold

$$xyx = y \text{ and } yxy = x$$

Definition1.2. A semigroup S is called quasi- seperative if for any $x,y \in S$, $x^2 = xy = y^2 \Rightarrow x = y$.
Definition1.3. A semigroup S is called weakly seperative if $x^2 = xy = yx = y^2$
$\Rightarrow x = y$ for all x,y in S.
Definition1.4. A semigroup S is called seperative if i) $x^2 = xy$ and $y^2 = yx \Rightarrow x = y$
ii) $x^2 = yx$ and $y^2 = xy \Rightarrow x = y$

II. PRELIMANARIES

Theorem2.1. Let (S, +, ·) be a semi ring and (S, ·) is a anti –inverse semigroup satisfying the identity a + 1 = 1 for all aϵs then(S, +) is an anti –inverse semigroup.

Poof: let (S, +, ·) be asemi ring and (S, ·) is a anti –inverse semigroup satisfying the identity a + 1 = 1 for all aϵS . Let for any aϵs there exist an element xϵs such that xax = a

Consider, x + a + x = x + a.1 + x = x + a (1 + xa) + x = x + a + axa + a = x + x + axa + a = x(1 + 1) + axa + a = x.1 + (ax + 1)a = x + 1.a = x + a = axa + a = (ax + 1)a = 1.a
= a. Hence, x + a + x = a
Similarly, a + x + a = x. Therefore, (S, +) is anti-inverse semigroup.

Theorem2.2. Let (S, +, ·) be LA-semi ring and (S, ·) is a anti –inverse semigroup then the product of two anti-inverse element is also anti-inverse element in (S, ·).

Proof : Let (S, +, ·) be LA-semi ring and (S, ·) is an anti –inverse semigroup
Let a,b are two elements in (S, ·) then their exist x, y elements in (S, ·) such that
xax = a and yby = b
Consider, yxabyx = bybxabyaxa = byaxbbyaxa = byaxabyba = byxyxa = byxaxy
= byay = ayby = ab. Hence, yxabyx = ab
Similarly, we can prove that baxyba = xy

Theorem2.3. Let (S, +, ·) be a LA-semi ring and (S, ·) is an anti –inverse semigroup then (S, .) is an abeliah semigroup.

Proof : Let (S, +, ·) be a LA-semi ring and (S, ·) is a anti –inverse semigroup From the above theorem 2, for any a, bϵ(S, ·) their exist x, y, bϵ(S, ·)
Such that yxabyx = ab $\Rightarrow$ yxaxyb = ab $\Rightarrow$ yayb = ab $\Rightarrow$ ybya = ab $\Rightarrow$ ba = ab
Hence (S, .) is an abelian semigroup

Theorem2.4. Let (S, +, ·) be a semi ring and (S, ·) is a anti –inverse semigroup satisfying the identity a + 1 = 1 for all aϵs then i) (S, +) is an abelian semigroup ii) The sum of two anti-inverse elements is again anti inverse element (S, +)

Proof: Let (S, +, ·) be a semi ring and (S, ·) is a anti –inverse semigroup satisfying the identity a + 1 = 1 for all aϵS. Then the semigroup (S, +) is anti-inverse semigroup from the theorem1. Let a,b in (S, +), then there exist x, y in (S, +) such that x + a + x = a and y + b + y = b and c + x = c = x, b + y + b = y
Consider, y + x + a + b + y + x = b+ y + b + x + a + b + y + a + x + a
= b + y+ a + x + b + b + y + a + x + a = b + y + a + x +a + b + y + b + x + a
= b + y + x + y + x + a = b + y + x + a + x + y = b + y + a + y = a + y + b + y = a + b

Therefore, y + x + a + b + y + x = a + b→(i)

Similarly, we can prove that b + a + x + y + b + a = x + y. Therefore, a + b is an anti-inverse element in (S, +). Therefore the sum of two anti-inverse elements is again anti-inverse elements in (S, +).

To show that (S, +) is an abelian semigroup: From (i), a + b = y + x + a + b + y + x = y + x + a + x + y + b = y + a + y + b = y + b + y + a = b + a. Hence, a + b = b + a.
Therefore, (S, +) is an abelian semigroup.

Theorem2.5. Let (S, +, ·) be a LA-semiring and (S, ·) is an anti inverse semigroup satisfying the identity a + 1 = 1, for all a in S then (S, +) is i) quasi-seperative ii) weakly seperative iii) seperative

Proof: Let (S, +, ·) be a LA-semiring and (S, ·) is an anti-inverse semigroup satisfying the identity a + 1 = 1 for all a in S then from theorem2.2, (S, +) is an anti-inverse semigroup

Let a, b ϵ (S, +) then consider a + a = a +b ⇒ a(1+1) = a + b ⇒ a.1 = a + yby = ab + b + yby = yxabyx + b + yby = yxaxyb + b + yby = (yxaxy + 1)b + yby = 1.b + yby = b + b
= b(1+1) = b.1. Hence, a = b
Therefore, a + a = a + b ⇒ a = b

Similarly, we can prove that a + b = b + b ⇒ a = b
Hence (S, +) is quasi- seperarive →(i)
From the theorem2.4, (S, +) is commutative, that is, a + b = b + a So a + a = a + b = b +a = b + b ⇒ a = b
Hence (S, +) is weakly seperative → (ii)
From the (i) and (ii) clearly, (S, +) is seperative.

Theorem2.6. Let (S, +, ·) be a LA-semiring and (S, ·) is an anti-inverse semigroup satisfying the identity a + 1 = 1 for all a in S then (S, +, ·) be a medial semiring.

Proof: Let (S, +, ·) be a LA-semiring and (S, ·) is an anti-inverse semigroup satisfying the identity a + 1 = 1 for all a in S
From the theorem2.3, (S, ·) is an abelian semigroup.
From the theorem2.4, (S, +) is an abelian semigroup.
Let a,b,c,d ϵ(S, ·) then abcd = a(bc)d = a(cb)d = acbd
Hence, abcd = acbd. Therefore, (S, ·) is a medial semigroup.
Similarly, (S, +) is also a medial semigroup. Hence, (S, +, ·) is a medial semiring.

Theorem2.7. Let (S, +, ·) be a LA-semiring and (S, ·) is an anti-inverse semigroup satisfying the identity a + 1 = 1 for all a in S then (S, +) ia an anti-inverse semigroup.

Proof: Let (S, +, ·) be a LA-semiring and (S, ·) is an anti-inverse semigroup satisfying the identity a + 1 = 1 for all a in S. Since, (S, ·) is an anti-inverse semigroup, for every aϵ(S, ·) there exist xϵ(S, ·) such that xax = a and axa = x.

Consider, x + a + x = x + xax + x = x(1+ ax) + x = xax + x = (xa + 1)x = xa.x ⇒
x + a + x = a.

Similarly, we can prove that a + x + a = x. Hence, a is an anti-inverse element of (S, +)
Therefore, (S, +) is an anti-inverse semigroup.

Theorem2.8. Let (S, +, ·) be a LA-semiring and (S, ·) is an anti-inverse semigroup satisfying the identity a + 1 = 1 for all a in S then the sum of two anti-inverse elements is also an anti-inverse element in (S, +).

Proof: Proof is similar to theorem4.

Theorem2.9. Let (S, ·) be an anti-inverse semigroup. If η is a relation defined on S by η = {(a, b) η ϵ S, $e_a a = e_b b$ where e_a, e_b unit elements of a, b respectively in S} then η is maximum 5-potent congruence on S.

Proof: Let (S, ·) be an anti-inverse semigroup. If η is a relation defined on S by η = {(a, b) η ϵ S, $e_a a = e_b b$ where e_a, e_b unit elements of a, b respectively in S}

First we show that η is an equalence relation on S. For any a in S, a = a ⇒ $a^5 = a^5$ ⇒ $e_a a = e_a a$ ⇒ a η a. Hence η is reflexive.

Let a η b and b η c ⇔ $e_a a = e_b b$ and $e_b b = e_c c$ so $e_a a = e_b b = e_c c$ ⇒ $e_a a = e_c c$ ⇔ a η c.
a η b and b η c ⇒ a η c . Hence η is transitive

$a\ \eta\ b \Leftrightarrow e_a a = e_b b \Leftrightarrow a^5 = b^5 \Leftrightarrow a = b \leftrightarrow b = a \Leftrightarrow b^5 = a^5 \Leftrightarrow e_b b = e_a a \Leftrightarrow b\ \eta\ a$. Hence η is symmetric. Therefore, η is an equalence relation.
Let $a\ \eta\ b\ e_a a = e_b b \Leftrightarrow a^5 = b^5 \Leftrightarrow a^5 z^5 = b^5 z^5 \Leftrightarrow (az)^5 = (bz)^5 \Leftrightarrow (az)^4(az) = (bz)^5(bz) \Leftrightarrow e_{az}az = e_{bz}bz \Leftrightarrow az\ \eta bz$. Hence η is right compatibility.
Let $a\ \eta\ b \Leftrightarrow e_a a = e_b b \Leftrightarrow a^5 = b^5 \Leftrightarrow z^5 a^5 = z^5 b^5 \Leftrightarrow (za)^5 = (zb)^5 \Leftrightarrow (za)^4(za) = (zb)^4(zb) \Leftrightarrow e_{za}za = e_{zb}zb \Leftrightarrow za\ \eta\ zb$. Hence η is left compatibility.
Therefore, η is compatability on S.
Let $a^5\ \eta\ b^5 \Leftrightarrow e_a a^5 = e_b b^5 \Leftrightarrow e_a.a = e_b.b \Leftrightarrow a\ \eta\ b$. Therefore, η is 5-potent congruence relation on S.
To prove η is maximum, let μ be any 5-potent congruence relation on S. Let $(a, b) \epsilon\ \mu \Leftrightarrow (a^5, b^5) \epsilon\ \mu \Leftrightarrow (ea, eb) \epsilon\ \mu$. We know that for all $(e_a, e_b) \epsilon\ \mu$ and $(a, b) \epsilon\ \mu \Rightarrow (e_a.a, e_b.b) \epsilon\ \mu$. Since $e_a.a = e_b.b \Leftrightarrow a\eta b \Leftrightarrow (a, b) \epsilon\ \eta$ and $(a, b) \epsilon\ \mu \Rightarrow \mu \leq \eta$. Hence η is maximum 5-potent congruence relation S.

Theorem2.10. Let (S, + .) be a LA-semiring and (S, .) be an anti-inverse semi group and let η be a congruence relation on S. Then S/ η is an anti-inverse sub semigroup.

Proof: Let (S,+ .) be a LA-semiring and (S, .) be an anti-inverse semi group and let η be a congruence relation on S.
Therefore we can construct the congruence class S/ η such that S/ η = {a η :aϵ(S, .)} where a η is a congruence class of a.
Define o on S/ η in the following way. For any aη, bηϵS/ η Such that (a η) o (a η) = (ab) η. Let $a\ \eta = a^1\ \eta$ and $b\ \eta = b^1\ \eta$ then $(a\ \eta)o(b\ \eta) = (ab)\ \eta \Rightarrow (a^1\ \eta)o(b^1\ \eta) = (ab)\ \eta \Rightarrow (a^1 b^1)\ \eta = (ab)\ \eta$. Hence o is well defined and it is associative. Hence (S/ η, .) is an anti-inverse sub semigroup.

Theorem2.11. Let η be a congruence relation on an anti-inverse semigroup S then η^n is also a congruence relation on S.
Proof: Let η be a congruence relation on an anti-inverse semigroup S
Let a ηb then there exist $t_1, t_2, t_3, t_{n-1}\ \epsilon S$ and by transitivity
We have $a\eta t_1, t_1\eta t_2, t_2\eta t_3,\ldots\ldots\ldots\ldots t_{n-1}\eta b \Rightarrow a\ \eta^n b$. it is easy to see that η^n is an equivalence relation. Let cϵS then c aηcb (Since η is compatible)
$ca\ \eta ct_1, ct_1\ \eta ct_2, ct_2\ \eta ct_3,\ldots\ldots\ldots\ldots ct_{n-1}\eta cb$. Hence $a\ \eta^n b \Rightarrow ca\ \eta^n cb$ similarly, we can prove that $a\ \eta^n b \Rightarrow ac\ \eta^n bc$.
Hence η^n is compatible. Therefore η^n is a congruence relation on S.

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